An Irrotational Flow Field That Approximates Flat Plate Boundary Conditions

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Abstract

A irrotational solution is derived for the steady-state Navier-Stokes equations that approximately satisfies the boundary conditions for flow over a finite flat plate. The nature of the flow differs substantially from boundary layer flow, with severe numerical difficulties in some regions.

An analytic function having the form

$$f(z) = v + iu, \tag{1}$$

leads to an exact solution of the two-dimensional steady-state Navier-Stokes equations, i.e.,

$$\begin{aligned} \rho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla p + \mu \nabla^2 \boldsymbol{u}; \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{2}$$

This occurs because (1) leads to a velocity field having the properties

$$\nabla^2 \boldsymbol{u} = \boldsymbol{0}; \tag{3}$$

and

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0; \\ \nabla \times \boldsymbol{u} &= \boldsymbol{0}. \end{aligned} \tag{4}$$

This is a subset of the generalized Beltrami flows[1], and note that (4) are the Cauchy-Riemann equations for (1). Substituting (3) and (4) into (2), and using the identity

$$\boldsymbol{u} \cdot \nabla\, \boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2} \nabla u^2 \tag{5}$$

leads to

$$\frac{\rho_0}{2}\nabla u^2 = -\nabla p, \tag{6}$$

or

$$\frac{p}{\rho_0} + \frac{u^2}{2} = C. \tag{7}$$

The main difficulty with (1) involves finding a function $f(z)$ satisfying useful no-slip boundary conditions. Consider the function

$$f(z) = \lim_{\epsilon\to 0}\frac{1}{L}\int_0^L \frac{\sin\left(2\pi\left(z-z_0\right)/L\right) i dx_0}{A + \left[B\sin\left(2\pi\left(z-z_0\right)/L\right) - A\right] e^{(z-z_0)^2/\epsilon}}. \tag{8}$$

Here $z = x + iy$, $z_0 = x_0 + iy_0$ and $y_0 = 0$. When $y \to \infty$,

$$f(z) \to \frac{1}{L}\int_0^L \frac{\sin\left(2\pi\left(z-z_0\right)/L\right) i dx_0}{A} = 0. \tag{9}$$

Making use of the identity

$$\begin{aligned} \sin\left(2\pi\left(z-z_0\right)/L\right) &= \cos\left(2\pi x_0/L\right)\sin\left(2\pi z/L\right) \\ &\quad - \sin\left(2\pi x_0/L\right)\cos\left(2\pi z/L\right), \end{aligned} \tag{10}$$

the terms $\cos\left(2\pi z/L\right)$ and $\sin\left(2\pi z/L\right)$ become large as $y \to \infty$; however, $\sin\left(2\pi x_0/L\right)$ and $\cos\left(2\pi x_0/L\right)$ are precisely zero when integrated over $0 \le x_0 \le L$.

When $y = 0$,

$$f(z) = \lim_{\epsilon\to 0}\frac{1}{L}\int_0^L \frac{\sin\left(2\pi\left(x-x_0\right)/L\right) i dx_0}{A + \left[B\sin\left(2\pi\left(x-x_0\right)/L\right) - A\right] e^{(x-x_0)^2/\epsilon}}. \tag{11}$$

When $x \neq x_0$, the denominator in (11) diverges. However, as $x \to x_0$, the denominator approaches $B\sin\left(2\pi\left(x-x_0\right)/L\right)$, so that as $\epsilon \to 0$, u approaches a rectangle function between $x = 0$ and $x = L$ and v approaches zero, meeting the boundary conditions for flow over a flat plate.

Although the integral (8) can be difficult to evaluate in some regions, software packages seem to have less trouble converging when $B >> A$, at least for $y = 0$. Choosing $\epsilon = 0.0001$, $A = 5.6 \times 10^{-4}$, $B = 100A$ and $L = 1\,\mathrm{m}$ satisfies the flat plate boundary condition as shown in figure 1. As $\epsilon \to 0$, u becomes more step-like at $x = 0$ and $x = 1$. Numerical approaches require ϵ to be finite, leading to regions at the plate edges where the boundary conditions are not satisfied. These regions can be made as small as required by reducing ϵ.

The boundary layer assumptions, i.e., $\frac{\partial^2 u}{\partial y^2} >> \frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial p}{\partial y} = 0$ are useful for analyzing flow over a flat plate, but they break down in the region at the leading edge[2], when $U_0 x/\nu \lesssim 10000$. This region of non-validity can be used to define an acceptable region where the flat plate boundary conditions are only satisfied

Figure 1: The u approximation to the flat plate boundary condition at $y = 0$ for $\epsilon = 0.0001$.

approximately by (11) and thus determines a practical value for ϵ. When $U_0 = 1\,\mathrm{m}\,/\,\mathrm{s}$ and $\nu = 10^{-6}\,\mathrm{m}^2\,/\,\mathrm{s}$, the extent of non-validity is approximately defined by $x \lesssim 0.01\,\mathrm{m}$. Setting $\epsilon = 0.00001$, $A = 5.9 \times 10^{-5}$, and $B = 100A$ leads to the boundary conditions being satisfied to within a tolerance on the order of 10^{-6} at the edges of the regions $-0.01 \leq x \leq 0.01$ and $0.99 \leq x \leq 1.01$ as shown in figure 2.

Although the flow described by (8) exactly satisfies the boundary conditions for a finite flat plate when $\epsilon \to 0$ (and approximately for small but finite ϵ), the flow field differs substantially from boundary layer flow. Figure 3 shows the regions having nonzero velocity components. As y increases, the integrand evolves into a sinusoidal-like function, until it becomes sufficiently sinusoidal that u and v both approach zero. Just below the upper "zero" line, small perturbations in the sinusoidal-like integrand functions lead to nonzero velocities. In some regions, either the first term or the second in the denominator in (8) dominate, allowing the other to be neglected. The latter is true when $y << 1$ above the plate, allowing the velocity field to be accurately approximated. When neither terms can be neglected, evaluation of the integral becomes tedious because of very large integrand magnitudes, leading software packages to either fail to converge or converge to incorrect solutions. Numerical determination of the velocity field for many regions remains a challenge.

Figure 2: The u approximation to the flat plate boundary condition at $y = 0$ for $\epsilon = 0.00001$.

References

1. Wang, C. Y. 1991. "Exact Solutions of the Steady-State Navier-Stokes Equations." *Annu. Rev. Fluid Mech.* **23**, 159-177.

2. Schlichting, H., "Boundary-Layer Theory", 6th ed., McGraw-Hill, 1968.

Figure 3: Upper and lower ”zero” lines. All nonzero velocity components are confined to the region between these lines.

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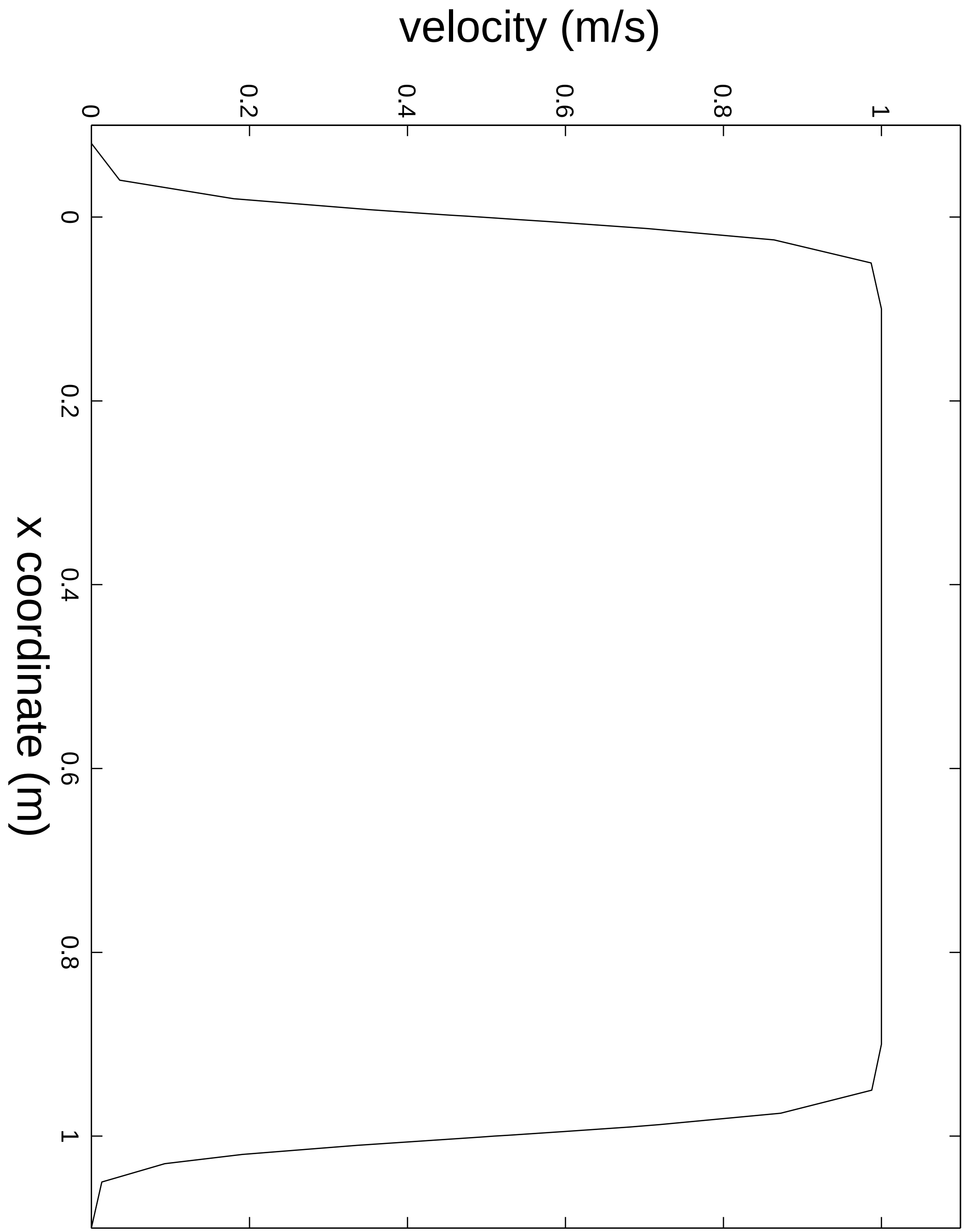

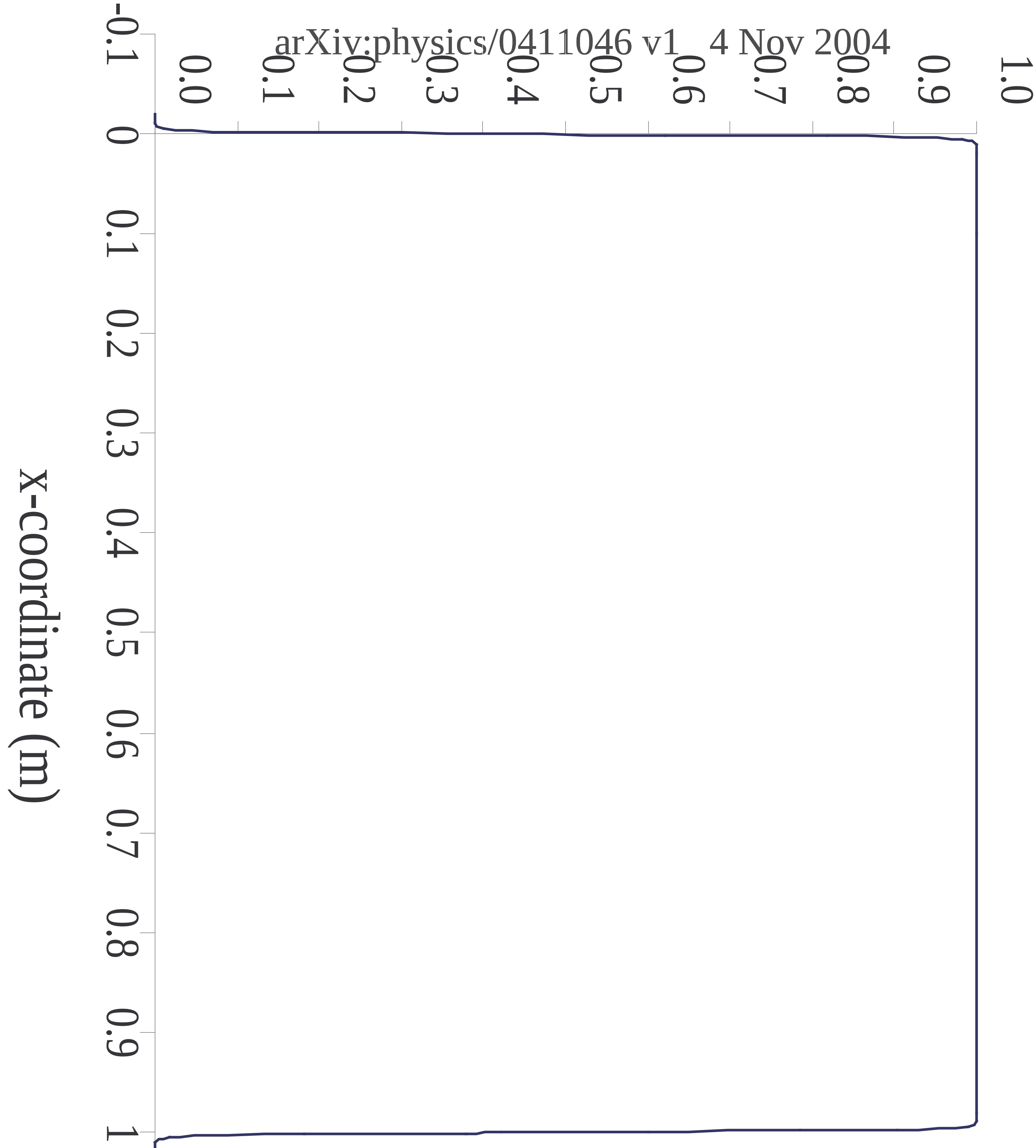
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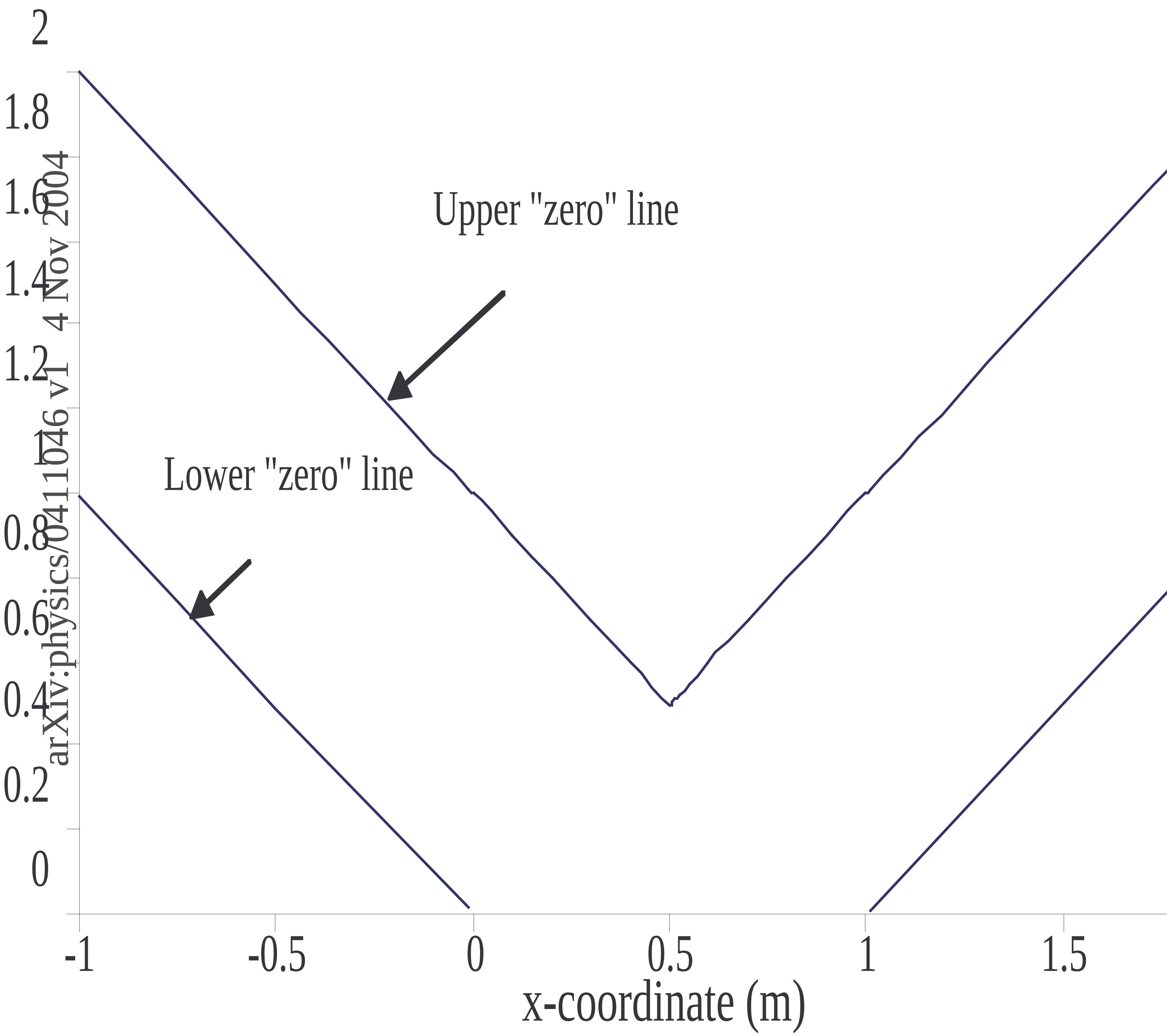
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